# 10. cvičení - řešení

Poznámka k racionální funkci $R$ u goniometrické substituce níže: často stačí jen určit podobu $R$ a pak ověřovat rovnosti $R(-u,v) = -R(u,v)$ apod. Když platí tyhle rovnosti, tak platí i odpovídající rovnici, kdy za $u$ volíme $\sin x$ a za $v$ volíme $\cos x$. Nicméně se teoreticky může stát, že rovnost $R(u,v) = -R(-u,v)$ splněna není, ale $R(\sin x, \cos x) = -R(-\sin x, \cos x)$. Zároveň mi přijde více polopatické/pochopitelnější ověřovat přímo vztah s těmi goniometrickými funkcemi.

**Příklad 1 (a)**
$\frac{1}{\sin x}$ je definováno pro $x \neq k\pi, k \in \mathbb{Z}$. Má smysl tedy integrovat jen na intervalech $(k\pi, (k+1)\pi), k \in \mathbb{Z}$.

**Jak přijít na vhodnou substituci:** Integrujeme funkci $\frac{1}{\sin x}$. Chceme použít postup z teorie, tedy na tuto funkci chceme nahlížet jako na racionální (polynom lomeno polynom) funkci o proměnných $\sin x$ a $\cos x$. Tj. chceme najít racionální funkci $R$ takovou, že $R(\sin x, \cos x) = \frac{1}{\sin x}$. Jaká racionální funkce to splňuje? Tato: $R(u,v) = \frac{1}{u}$. Nyní chci zjistit, který z případů v teorii je splněn - tj. podívám se postupně na $R(-\sin x, \cos x), R(\sin x, -\cos x)$ a $R(-\sin x, -\cos x)$ a popřemýšlím nad tím, jestli je splněn nějaký příklad z rozboru v teorii.

| Výraz | Čemu je roven | Čemu chci, aby se to rovnalo | Platí daný vztah? |
|---|---|---|---|
| $R(-\sin x, \cos x)$ | $\frac{1}{-\sin x}$ | $-\frac{1}{\sin x}$ | ano |

**Poznámka k výpočtu níže:** Došli jsme k tomu, že chceme použít substituci $y = \cos x$, $\mathrm{d}y = -\sin x \, \mathrm{d}x$. Potřebujeme v integrálu tedy najít $-\sin x \, \mathrm{d}x$. Bohužel to tam nemáme, takže si to tam musíme vyrobit - zde tak, že výraz přenásobíme jedničkou $\frac{-\sin x}{-\sin x}$. Tím jsme vyrobili, co jsme potřebovali. Dole nám vzniklo $-\sin^2 x$ a to potřebujeme vyjádřit pomocí $y$. My ale umíme substituovat jen $\cos x$, takže potřebujeme to $-\sin^2 x$ nějak přetvořit na něco, kde se objevuje jen kosinus a čísla. Využijeme proto Pythagorovu větu $1 = \sin^2 x + \cos^2 x$.

$$\int \frac{1}{\sin x} \, \mathrm{d}x = |y = \cos x, \ \mathrm{d}y = -\sin x \, \mathrm{d}x| = \int \frac{-\sin x}{-\sin^2 x} \, \mathrm{d}x = \int \frac{-\sin x}{-(1-\cos^2 x)} \, \mathrm{d}x =$$
$$= \int \frac{1}{y^2 - 1} \, \mathrm{d}y$$

Parciální zlomky: $\frac{1}{y^2-1} = \frac{A}{y-1} + \frac{B}{y+1} \implies 1 = Ay + A + By - B \implies A + B = 0, A - B = 1$

$$\int \frac{1}{y^2-1} \, \mathrm{d}y \overset{\text{lin.}}{=} -\frac{1}{2} \int \frac{1}{y+1} \, \mathrm{d}y + \frac{1}{2} \int \frac{1}{y-1} \, \mathrm{d}y \overset{c}{=} -\frac{1}{2} \log|y+1| + \frac{1}{2} \log|y-1|$$

$$\implies \int \frac{1}{\sin x} \, \mathrm{d}x \overset{c}{=} \frac{1}{2} \log|\cos x - 1| - \frac{1}{2} \log|\cos x + 1|$$

**Příklad 1 (b)**
Zřejmě je vnitřek integrálu definován pro všechna $x \in \mathbb{R}$.

**Výběr substituce:**
Chci najít racionální fci $R$ t.ž. $R(\sin x, \cos x) = \frac{1}{\sin^4 x + \cos^4 x}$. To je fce $R(u,v) = \frac{1}{u^4+v^4}$.

| Výraz | Čemu je roven | Čemu chci, aby se to rovnalo | Platí daný vztah? |
|---|---|---|---|
| $R(-\sin x, \cos x)$ | $\frac{1}{(-\sin x)^4+\cos^4 x}$ | $-\frac{1}{\sin^4 x+\cos^4 x}$ | ne |
| $R(\sin x, -\cos x)$ | $\frac{1}{\sin^4 x+(-\cos x)^4}$ | $-\frac{1}{\sin^4 x+\cos^4 x}$ | ne |
| $R(-\sin x, -\cos x)$ | $\frac{1}{(-\sin x)^4+(-\cos x)^4}$ | $\frac{1}{\sin^4 x+\cos^4 x}$ | ano |

**Ověření vzorců pro substituci uvedeného v teorii:**

$$\begin{aligned}
\tan^2 x &= \frac{\sin^2 x}{\cos^2 x}\\
\tan^2 x + 1 &= \frac{1}{\cos^2 x}\\
\cos^2 x &= \frac{1}{\tan^2 x + 1}\\
\sin^2 x &= 1 - \cos^2 x = 1 - \frac{1}{\tan^2 x + 1} = \frac{\tan^2 x + 1 - 1}{\tan^2 x + 1} = \frac{\tan^2 x}{\tan^2 x + 1}
\end{aligned}$$

$$\int \frac{1}{\sin^4 x + \cos^4 x}\,\mathrm{d}x = \left| y = \tan x,\ \mathrm{d}y = \frac{1}{\cos^2 x}\,\mathrm{d}x \right| = \int \frac{\cos^2 x}{\sin^4 x + \cos^4 x} \cdot \frac{1}{\cos^2 x}\,\mathrm{d}x =$$
$$= \int \frac{\frac{1}{y^2+1}}{\left(\frac{y^2}{y^2+1}\right)^2 + \left(\frac{1}{y^2+1}\right)^2}\,\mathrm{d}y = \int \frac{1}{y^2+1} \cdot \frac{(y^2+1)^2}{y^4+1}\,\mathrm{d}y = \int \frac{y^2+1}{y^4+1}\,\mathrm{d}y$$

**Vyýpočet:**
Rozklad $y^4 + 1$ pomocí finty z reciprokých rovnic.
Vizte: https://www2.karlin.mff.cuni.cz/˜ portal/komplexni_cisla.dp/?page=rovnice-reciproke

$$\begin{aligned}
&y^4 + 1 = y^2\left(y^2 + \frac{1}{y^2}\right)\\
&z = y + \frac{1}{y} \implies z^2 = y^2 + 2 + \frac{1}{y^2}\\
&\implies y^2 + \frac{1}{y^2} = z^2 - 2 = \left(z - \sqrt{2}\right)\left(z + \sqrt{2}\right)\\
&\implies y^4 + 1 = y^2(y^2 + \frac{1}{y^2}) = y^2\left(z - \sqrt{2}\right)\left(z + \sqrt{2}\right) = y\left(y + \frac{1}{y} - \sqrt{2}\right) y \left(y + \frac{1}{y} + \sqrt{2}\right) =\\
&\qquad = \left(y^2 - y\sqrt{2} + 1\right)\left(y^2 + y\sqrt{2} + 1\right)
\end{aligned}$$

Trojčleny $y^2 \pm y\sqrt{2} + 1$ již nelze rozložit (záporný diskriminant).
Parciální zlomky:

$$\begin{aligned}
&\frac{y^2+1}{(y^2 - y\sqrt{2} + 1)\,(y^2 + y\sqrt{2} + 1)} = \frac{Ay + B}{(y^2 + y\sqrt{2} + 1)} + \frac{Cy + D}{(y^2 - y\sqrt{2} + 1)}\\
&y^2 + 1 = (Ay + B)\left(y^2 - y\sqrt{2} + 1\right) + (Cy + D)\left(y^2 + y\sqrt{2} + 1\right)\\
&y^2 + 1 = Ay^3 - Ay^2\sqrt{2} + Ay + By^2 - By\sqrt{2} + B + Cy^3 + Cy^2\sqrt{2} + Cy + Dy^2 + Dy\sqrt{2} + D\\
&y^2 + 1 = y^3(A + C) + y^2\left(-A\sqrt{2} + B + C\sqrt{2} + D\right) + y\left(A - B\sqrt{2} + C + D\sqrt{2}\right) + B + D
\end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}
&y^3 \implies 0 = A + C \implies -A = C \overset{2}{\implies} A = 0\\
&y^2 \implies 1 = -A\sqrt{2} + B + C\sqrt{2} + D \overset{1}{\implies} 1 = 2C\sqrt{2} + B + D \overset{4}{\implies} C = 0\\
&y \implies 0 = A - B\sqrt{2} + C + D\sqrt{2} \overset{1+2}{\implies} 0 = \sqrt{2}(D - B) \implies B = D\\
&y^0 \implies 1 = B + D \overset{4}{\implies} B = D = \frac{1}{2}
\end{aligned}$$

Máme tedy: $A = C = 0, B = D = \frac{1}{2}$.

$$\begin{aligned}
\int \frac{y^2+1}{y^4+1}\,\mathrm{d}y &\overset{\text{lin.}}{=} \frac{1}{2}\int \frac{1}{y^2 + y\sqrt{2} + 1}\,\mathrm{d}y + \frac{1}{2}\int \frac{1}{y^2 - y\sqrt{2} + 1}\,\mathrm{d}y =\\
&= \frac{1}{2}\int \frac{1}{\left(y + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}}\,\mathrm{d}y + \frac{1}{2}\int \frac{1}{\left(y - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}}\,\mathrm{d}y =\\
&= \int \frac{1}{\left(\frac{y + \frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right)^2 + 1}\,\mathrm{d}y + \int \frac{1}{\left(\frac{y - \frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right)^2 + 1}\,\mathrm{d}y =\\
&= \left| z = \sqrt{2}\left(y + \frac{\sqrt{2}}{2}\right),\ \mathrm{d}z = \sqrt{2}\,\mathrm{d}y, u = \sqrt{2}\left(y - \frac{\sqrt{2}}{2}\right),\ \mathrm{d}u = \sqrt{2}\,\mathrm{d}y \right| =\\
&= \frac{1}{\sqrt{2}}\int \frac{1}{z^2+1}\,\mathrm{d}z + \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\,\mathrm{d}u \overset{\text{c}}{=} \frac{1}{\sqrt{2}}\arctan z + \frac{1}{\sqrt{2}}\arctan u =\\
&\overset{\text{c}}{=} \frac{1}{\sqrt{2}}\arctan\sqrt{2}\left(y + \frac{\sqrt{2}}{2}\right) + \frac{1}{\sqrt{2}}\arctan\sqrt{2}\left(y - \frac{\sqrt{2}}{2}\right) =\\
&\overset{\text{c}}{=} \frac{1}{\sqrt{2}}\arctan\sqrt{2}\left(\tan x + \frac{\sqrt{2}}{2}\right) + \frac{1}{\sqrt{2}}\arctan\sqrt{2}\left(\tan x - \frac{\sqrt{2}}{2}\right) =\\
&\overset{\text{c}}{=} \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x + 1\right) + \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x - 1\right)
\end{aligned}$$

**Příklad 1 (c)** $\int \frac{\cos^3 x}{2-\sin x}\,\mathrm{d}x$
Podmínky: $\sin x \neq 2$ - platí vždy.

**Jak zvolit substituci:**
Chci $R(\sin x, \cos x) = \frac{\cos^3 x}{2-\sin x}$. Tj. máme $R(u,v) = \frac{v^3}{1-u}$.

| Výraz | Čemu je roven | Čemu chci, aby se to rovnalo | Platí daný vztah? |
|---|---|---|---|
| $R(-\sin x, \cos x)$ | $\frac{\cos^3 x}{1-(-\sin x)}$ | $-\frac{\cos^3 x}{1-\sin x}$ | ne |
| $R(\sin x, -\cos x)$ | $\frac{(-\cos x)^3}{1-\sin x}$ | $-\frac{\cos^3 x}{1-\sin x}$ | ano |

$$\int \frac{\cos^3 x}{2-\sin x}\,\mathrm{d}x = |y = \sin x,\ \mathrm{d}y = \cos x\,\mathrm{d}x| = \int \frac{(1-\sin^2 x)\cos x}{2-\sin x}\,\mathrm{d}x = \int \frac{1-y^2}{2-y}\,\mathrm{d}y =$$

$$\int \frac{y^2-1}{y-2}\,\mathrm{d}y = \int y + 2 + \frac{3}{y-2}\,\mathrm{d}y = |z = y-2,\ \mathrm{d}z = \ \mathrm{d}y| \overset{\text{lin.}}{=} \frac{y^2}{2} + 2y + 3\int \frac{1}{z}\,\mathrm{d}z =$$

$$\overset{\text{c}}{=} \frac{y^2}{2} + 2y + 3\log|z| \overset{\text{c}}{=} \frac{y^2}{2} + 2y + 3\log|y-2| \overset{\text{c}}{=} \frac{\sin^2 x}{2} + 2\sin x + 3\log|\sin x - 2|$$

**Příklad 1 (d)** $\int \frac{1}{2-\cos x}\ \mathrm{d}x$
Všimněme si, že $\frac{1}{2-\cos x}$ je definováno všude.

**Jak přijít na substituci:**
Chceme $R(\sin x, \cos x) = \frac{1}{2-\cos x}$. Tj. $R(u,v) = \frac{1}{2-v}$.

| Výraz | Čemu je roven | Čemu chci, aby se to rovnalo | Platí daný vztah? |
|---|---|---|---|
| $R(-\sin x, \cos x)$ | $\frac{1}{2-\cos x}$ | $-\frac{1}{2-\cos x}$ | ne |
| $R(\sin x, -\cos x)$ | $\frac{1}{2-(-\cos x)}$ | $-\frac{1}{2-\cos x}$ | ne |
| $R(-\sin x, -\cos x)$ | $\frac{1}{2-(-\cos x)}$ | $\frac{1}{2-\cos x}$ | ne |

Nenastává ani jeden z prvních tří případů, musíme proto volit subtituci $y = \tan\frac{x}{2}$.

**Ověření vzorců pro subsituci:**

$$t = \tan\frac{x}{2}$$

$$\tan^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} \implies 1+\tan^2\frac{x}{2} = 1 + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1}{\cos^2\frac{x}{2}} \qquad \text{(Pythagorova věta)}$$

$$\implies \cos^2\frac{x}{2} = \frac{1}{1+\tan^2\frac{x}{2}}$$

$$\mathrm{d}t = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\ \mathrm{d}x = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)\ \mathrm{d}x = \frac{1}{2}\left(1+t^2\right)\ \mathrm{d}x \implies \frac{2}{1+t^2}\ \mathrm{d}t = \ \mathrm{d}x$$

$$\sin x = \sin\left(2\frac{x}{2}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2} = 2\tan\frac{x}{2}\cdot\frac{1}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \cos\left(2\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2} - \left(1-\cos^2\frac{x}{2}\right) = \frac{1}{1+t^2} - \left(1 - \frac{1}{1+t^2}\right) =$$

$$= \frac{1}{1+t^2} - \frac{1+t^2-1}{1+t^2} = \frac{1-t^2}{1+t^2}$$

**Výpočet:**

$$\int \frac{1}{2-\cos x}\ \mathrm{d}x = \left| t = \tan\frac{x}{2}, \frac{2}{1+t^2}\ \mathrm{d}t = \ \mathrm{d}x \right| = \int \frac{1}{2 - \frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\ \mathrm{d}t = \int \frac{1}{\frac{2+2t^2-1+t^2}{1+t^2}}\frac{2}{1+t^2}\ \mathrm{d}t =$$

$$= \int \frac{1+t^2}{1+3t^2}\frac{2}{1+t^2}\ \mathrm{d}t = \int \frac{2}{(\sqrt{3}t)^2+1}\ \mathrm{d}t = \left| y = \sqrt{3}t,\ \mathrm{d}y = \sqrt{3}\ \mathrm{d}t \right| =$$

$$\overset{\text{lin.}}{=} \frac{2}{\sqrt{3}}\int \frac{1}{y^2+1}\ \mathrm{d}y \overset{\text{c}}{=} \frac{2}{\sqrt{3}}\arctan y \overset{\text{c}}{=} \frac{2}{\sqrt{3}}\arctan\left(\sqrt{3}t\right) \overset{\text{c}}{=} \frac{2}{\sqrt{3}}\arctan\left(\sqrt{3}\tan\frac{x}{2}\right)$$

**Příklad 1 (e)** $\int \frac{1}{1+\sin^2 x}\ \mathrm{d}x$

**Výběr substituce:**
Chci: $R(\sin x, \cos x) = \frac{1}{1+\sin^2 x}$. Tj. $R(u,v) = \frac{1}{1+u^2}$.

| Výraz | Čemu je roven | Čemu chci, aby se to rovnalo | Platí daný vztah? |
|---|---|---|---|
| $R(-\sin x, \cos x)$ | $\frac{1}{1+(\sin x)^2}$ | $-\frac{1}{1+\sin^2 x}$ | ne |
| $R(\sin x, -\cos x)$ | $\frac{1}{1+\sin^2 x}$ | $-\frac{1}{1+\sin^2 x}$ | ne |
| $R(-\sin x, -\cos x)$ | $\frac{1}{1+(-\sin x)^2}$ | $\frac{1}{1+\sin^2 x}$ | ano |

Volme tedy substituci $t = \tan x$.

$$\int \frac{1}{1+\sin^2 x}\,\mathrm{d}x = \left| t = \tan x,\ \mathrm{d}x = \frac{1}{1+t^2}\,\mathrm{d}t \right| = \int \frac{1}{1+\frac{t^2}{1+t^2}} \cdot \frac{1}{1+t^2}\,\mathrm{d}t = \int \frac{t^2+1}{1+t^2+t^2}\frac{1}{1+t^2}\,\mathrm{d}t =$$
$$= \frac{1}{1+2t^2}\,\mathrm{d}t = \int \frac{1}{1+\left(t\sqrt{2}\right)^2}\,\mathrm{d}t = \left| y = t\sqrt{2},\ \mathrm{d}y = \sqrt{2}\,\mathrm{d}t \right| \overset{\text{lin.}}{=} \frac{1}{\sqrt{2}} \int \frac{1}{1+y^2}\,\mathrm{d}y =$$
$$\overset{c}{=} \frac{1}{\sqrt{2}} \arctan y \overset{c}{=} \frac{1}{\sqrt{2}} \arctan(\sqrt{2}t) \overset{c}{=} \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}\tan x\right)$$

**Příklad 1 (f)** $\int \frac{1}{\cos x \cdot \sin^3 x}\,\mathrm{d}x$

Podmínky: $\cos x \neq 0 \wedge \sin x \neq 0 \implies x \neq k\frac{\pi}{2}, k \in \mathbb{Z}$.

**Výběr substituce:**
Chci: $R(\sin x, \cos x) = \frac{1}{\cos x \cdot \sin^3 x}$. Tj. $R(u, v) = \frac{1}{v \cdot u^3}$.

| Výraz | Čemu je roven | Čemu chci, aby se to rovnalo | Platí daný vztah? |
|---|---|---|---|
| $R(-\sin x, \cos x)$ | $\frac{1}{\cos x \cdot (-\sin x)^3}$ | $-\frac{1}{\cos x \cdot \sin^3 x}$ | ano |

$$\int \frac{1}{\cos x \cdot \sin^3 x}\,\mathrm{d}x = |t = \cos x,\ \mathrm{d}t = -\sin x\,\mathrm{d}x| = \int \frac{-\sin x}{-\cos x \cdot \sin^4 x}\,\mathrm{d}x =$$
$$= \int \frac{-\sin x}{-\cos x \cdot (1-\cos^2 x)^2}\,\mathrm{d}x = \int \frac{-1}{t \cdot (1-t^2)^2}\,\mathrm{d}t$$

Parciální zlomky:

$$\frac{-1}{t(t^2-1)^2}=\frac{-1}{t(t+1)^2(t-1)^2}=\frac{A}{t}+\frac{B}{t+1}+\frac{C}{(t+1)^2}+\frac{D}{t-1}+\frac{E}{(t-1)^2}$$

$$-1=A(t^2-1)^2+Bt(t+1)(t-1)^2+Ct(t-1)^2+Dt(t-1)(t+1)^2+Et(t+1)^2$$

$$t=0\implies -1=A(-1)^2\implies A=-1$$

$$t=1\implies -1=E4\implies E=-\frac{1}{4}$$

$$t=-1\implies -1=C(-1)(-2)^2\implies C=\frac{1}{4}$$

$$-1=-(t^2-1)^2+Bt(t+1)(t-1)^2+\frac{1}{4}t(t-1)^2+Dt(t-1)(t+1)^2-\frac{1}{4}t(t+1)^2$$

$$-1=-t^4+2t^2-1+Bt^4+\left(B+\frac{1}{4}\right)t^3-2Bt^3-2\left(B+\frac{1}{4}\right)t^2+Bt^2+\left(B+\frac{1}{4}\right)t+Dt^4+$$
$$+\left(-\frac{1}{4}-D\right)t^3+2Dt^3+2\left(-\frac{1}{4}-D\right)t^2+Dt^2+\left(-\frac{1}{4}-D\right)t$$

$$-1=t^4(-1+B+D)+t^3\left(B+\frac{1}{4}-2B-\frac{1}{4}-D+2D\right)+$$
$$+t^2\left(2-2B-2\frac{1}{4}+B-2\frac{1}{4}-2D+D\right)+t\left(B+\frac{1}{4}-\frac{1}{4}-D\right)-1$$

Porovnáním koeficientů dostáváme soustavu:

$$-1+B+D=0\implies B+D=1\overset{2}{\implies}2D=\frac{1}{2}\implies B=\frac{1}{2}=D$$
$$-B+D=0\implies B=D$$
$$1-B-D=0$$
$$B-D=0$$

Řešení soustavy je: $A=-1,B=\frac{1}{2},C=\frac{1}{4},D=\frac{1}{2},E=-\frac{1}{4}$

$$\int\frac{-1}{t(1-t^2)^2}\,\mathrm{d}t=\int-\frac{1}{t}+\frac{1}{2}\frac{1}{t+1}+\frac{1}{4}\frac{1}{(t+1)^2}+\frac{1}{2}\frac{1}{t-1}-\frac{1}{4}\frac{1}{(t-1)^2}\,\mathrm{d}t=$$
$$=|\text{substituce}|\overset{c}{=}-\log|t|+\frac{1}{2}\log|t+1|-\frac{1}{4}\frac{1}{t+1}+\frac{1}{2}\log|t-1|+\frac{1}{4}\frac{1}{t-1}=$$
$$\overset{c}{=}\frac{1}{4}\frac{t+1-(t-1)}{(t-1)(t+1)}+\frac{1}{2}\log|(t+1)(t-1)|-\log|t|=$$
$$\overset{c}{=}\frac{1}{4}\frac{2}{t^2-1}+\frac{1}{2}\left(\log|t^2-1|-\log|t^2|\right)\overset{c}{=}\frac{1}{2(t^2-1)}+\frac{1}{2}\log\left|\frac{t^2-1}{t^2}\right|=$$
$$\overset{c}{=}\frac{1}{2(\cos^2x)-1}+\log\sqrt{\frac{1-\cos^2x}{\cos^2x}}\overset{c}{=}-\frac{1}{2\sin^2x}+\log\sqrt{\frac{\sin^2x}{\cos^2x}}=$$
$$\overset{c}{=}-\frac{1}{2\sin^2x}+\log|\tan x|$$

**Příklad 1 (g)** $\int\frac{\sin x}{1+\sin x}\,\mathrm{d}x$
Podmínky: $\sin x\neq-1\implies x\neq\frac{3\pi}{2}+k2\pi,k\in\mathbb{Z}$.

**Výběr substituce:**
Chci: $R(\sin x, \cos x) = \frac{\sin x}{1+\sin x}$. Tj. $R(u,v) = \frac{u}{1+u}$.

| Výraz | Čemu je roven | Čemu chci, aby se to rovnalo | Platí daný vztah? |
|---|---|---|---|
| $R(-\sin x, \cos x)$ | $\frac{-\sin x}{1-\sin x}$ | $-\frac{\sin x}{1+\sin x}$ | ne |
| $R(\sin x, -\cos x)$ | $\frac{\sin x}{1+\sin x}$ | $-\frac{\sin x}{1+\sin x}$ | ne |
| $R(-\sin x, -\cos x)$ | $\frac{-\sin x}{1-\sin x}$ | $\frac{\sin x}{1+\sin x}$ | ne |

$$\int \frac{\sin x}{1+\sin x}\,\mathrm{d}x = \left| t = \tan\frac{x}{2},\ \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t \right| = \int \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}} \frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{4t}{(1+t^2)^2} \cdot \frac{1+t^2}{1+t^2+2t}\,\mathrm{d}t =$$
$$= \int \frac{4t}{(t^2+1)(t+1)^2}\,\mathrm{d}t$$

Parciální zlomky:

$$\frac{4t}{(t^2+1)(t+1)^2} = \frac{At+B}{t^2+1} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$
$$4t = (At+B)(t^2+2t+1) + C(t+1)(t^2+1) + D(t^2+1)$$
$$t = -1 \implies -4 = 2D \implies D = -2$$
$$4t = At^3 + 2At^2 + At + Bt^2 + 2Bt + B + Ct^3 + Ct + Ct^2 + C - 2t^2 + -2$$

Porovnáním koeficientů dostáváme soustavu:

$$0 = A + C \implies C = -A \stackrel{4}{\implies} C = 0$$
$$0 = 2A + B + C - 2 \stackrel{1}{\implies} 2 = A + B \implies B = 2 - A \stackrel{4}{\implies} B = 2$$
$$4 = A + 2B + C \stackrel{2}{\implies} 4 = 2 + B + C$$
$$0 = B + C - 2 \stackrel{1}{\implies} 2 = B - A \stackrel{2}{\implies} 2 = 2 - 2A \implies A = 0$$

$$\int \frac{4t}{(t^2+1)(t+1)^2}\,\mathrm{d}t \stackrel{\text{lin.}}{=} \int \frac{2}{t^2+1}\,\mathrm{d}t + \int \frac{-2}{(t+1)^2}\,\mathrm{d}t = |y = t+1,\ \mathrm{d}y = \ \mathrm{d}t| =$$
$$= 2\arctan t - 2\int \frac{1}{y^2}\,\mathrm{d}y \stackrel{\text{c}}{=} 2\arctan t + 2\frac{1}{y} =$$
$$\stackrel{\text{c}}{=} 2\arctan t + 2\frac{1}{t+1} \stackrel{\text{c}}{=} 2\arctan\left(\tan\frac{x}{2}\right) + 2\frac{1}{\tan\frac{x}{2}+1} \stackrel{\text{c}}{=} 2\frac{x}{2} + \frac{2}{\tan\frac{x}{2}+1} =$$
$$\stackrel{\text{c}}{=} x + \frac{2}{\tan\frac{x}{2}+1}$$

**Příklad 1 (h)** $\int \frac{2-\sin x}{2+\cos x}\,\mathrm{d}x$
$\frac{2-\sin x}{2+\cos x}$ je definováno všude.

**Výběr substituce:**
Chci: $R(\sin x, \cos x) = \frac{2-\sin x}{2+\cos x}$. Tj. $R(u,v) = \frac{2-u}{2+v}$.

| Výraz | Čemu je roven | Čemu chci, aby se to rovnalo | Platí daný vztah? |
|---|---|---|---|
| $R(-\sin x, \cos x)$ | $\frac{2-(-\sin x)}{2+\cos x}$ | $-\frac{2-\sin x}{2+\cos x}$ | ne |
| $R(\sin x, -\cos x)$ | $\frac{2-\sin x}{2+(-\cos x)}$ | $-\frac{2-\sin x}{2+\cos x}$ | ne |
| $R(-\sin x, -\cos x)$ | $\frac{2-(-\sin x)}{2+(-\cos x)}$ | $\frac{2-\sin x}{2+\cos x}$ | ne |

$$\int \frac{2-\sin x}{2+\cos x}\ \mathrm{d}x = \left| t = \tan\frac{x}{2},\ \mathrm{d}x = \frac{2}{1+t^2}\ \mathrm{d}t \right| = \int \frac{2-\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\ \mathrm{d}t = \int \frac{\frac{2+2t^2-2t}{1+t^2}}{\frac{2+2t^2+1-t^2}{1+t^2}} \frac{2}{1+t^2}\ \mathrm{d}t =$$

$$= \int \frac{2t^2-2t+2}{1+t^2} \cdot \frac{1+t^2}{t^2+3} \cdot \frac{2}{1+t^2}\ \mathrm{d}t = \int \frac{2(2t^2-2t+2)}{(t^2+1)(t^2+3)}\ \mathrm{d}t$$

Parciální zlomky:

$$\begin{aligned}
&\frac{4t^2-4t+4}{(t^2+1)(t^2+3)} = \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+3} \\
&4t^2-4t+4 = (At+B)(t^2+3) + (Ct+D)(t^2+1) \\
&4t^2-4t+4 = At^3+3At+Bt^2+3B+Ct^3+Ct+Dt^2+D
\end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}
&0 = A + C \implies C = -A \overset{3}{\implies} C = 2 \\
&4 = B + D \overset{4}{\implies} D = 4 \\
&-4 = 3A + C \overset{1}{\implies} -4 = 2A \implies A = -2 \\
&4 = 3B + D \overset{2}{\implies} 0 = 2B \implies B = 0
\end{aligned}$$

$$\begin{aligned}
\int \frac{4t^2-4t+4}{(t^2+1)(t^2+3)}\ \mathrm{d}t &\overset{\text{lin.}}{=} \int \frac{-2t}{t^2+1}\ \mathrm{d}t + \int \frac{2t+4}{t^2+3}\ \mathrm{d}t = \\
&= \left| y = t^2+1,\ \mathrm{d}y = 2t\ \mathrm{d}t, z = t^2+3,\ \mathrm{d}z = 2t\ \mathrm{d}t \right| = \\
&= -\int \frac{1}{y}\ \mathrm{d}y + \int \frac{1}{z}\ \mathrm{d}z + 4\int \frac{1}{t^2+3}\ \mathrm{d}t = \left| u = \frac{t}{\sqrt{3}},\ \mathrm{d}u = \frac{1}{\sqrt{3}}\ \mathrm{d}t \right| = \\
&= -\log|y| + \log|z| + 4\sqrt{3}\int \frac{1}{u^2+1}\ \mathrm{d}u = \\
&\overset{c}{=} -\log|t^2+1| + \log|t^2+3| + 4\sqrt{3}\arctan u \overset{c}{=} \log\frac{t^2+3}{t^2+1} + 4\sqrt{3}\arctan\frac{t}{\sqrt{3}} = \\
&\overset{c}{=} \log\frac{\tan^2\frac{x}{2}+3}{\tan^2\frac{x}{2}+1} + 4\sqrt{3}\arctan\frac{\tan\frac{x}{2}}{\sqrt{3}}
\end{aligned}$$

**Příklad 1 (i)** $\int \frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x}\ \mathrm{d}x$
Vnitřek integrálu je definován všude.

**Výběr substituce:**

Chci: $R(\sin x, \cos x) = \frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x}$. Tj. $R(u,v) = \frac{u^3}{1+4v^2+3u^2}$.

| Výraz | Čemu je roven | Čemu chci, aby se to rovnalo | Platí daný vztah? |
|---|---|---|---|
| $R(-\sin x, \cos x)$ | $\frac{(-\sin x)^3}{1+4\cos^2 x+3(-\sin x)^2}$ | $-\frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x}$ | ano |

Z Pythagorovy věty plyne první rovnost níže.

$$\begin{aligned}\int \frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x}\,\mathrm{d}x &= \int \frac{\sin^3 x}{4+\cos^2 x}\,\mathrm{d}x = |t=\cos x,\ \mathrm{d}t = -\sin x\,\mathrm{d}x| = \int \frac{-(1-t^2)}{4+t^2}\,\mathrm{d}t = \\ &= \int \frac{t^2-1}{t^2+4}\,\mathrm{d}t = \int \frac{t^2-1+5-5}{t^2+4}\,\mathrm{d}t \overset{\text{lin.}}{=} \int 1\,\mathrm{d}t - 5\int \frac{1}{t^2+4}\,\mathrm{d}t = \\ &= t - \frac{5}{4}\int \frac{1}{\left(\frac{t}{2}\right)^2+1}\,\mathrm{d}t = \left|y=\frac{t}{2},\ \mathrm{d}y = \frac{1}{2}\,\mathrm{d}t\right| = t - \frac{5}{4}\cdot 2\int \frac{1}{y^2+1}\,\mathrm{d}y = \\ &\overset{c}{=} t - \frac{5}{2}\arctan y \overset{c}{=} t - \frac{5}{2}\arctan\frac{t}{2} \overset{c}{=} \cos x - \frac{5}{2}\arctan\frac{\cos x}{2}\end{aligned}$$

**Příklad 1 (j)** $\int \frac{3\sin^2 x+\cos^2 x}{\sin^2 x+3\cos^2 x}\,\mathrm{d}x$
Vnitřek integrálu je zřejmě definován všude.

**Výběr substituce:**
Chci: $R(\sin x, \cos x) = \frac{3\sin^2 x+\cos^2 x}{\sin^2 x+3\cos^2 x}$. Tj. $R(u,v) = \frac{3u^2+v^2}{u^2+3v^2}$.
Vidíme, že je splněn vztah $R(-\sin x, -\cos x) = R(\sin x, \cos x)$.
Proto volíme substituci $t = \tan x$. Integrujeme tedy jen na intervalech $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi), k \in \mathbb{Z}$.

Níže první rovnost plyne z Pythagorovy věty.

$$\begin{aligned}\int \frac{3\sin^2 x+\cos^2 x}{\sin^2 x+3\cos^2 x}\,\mathrm{d}x &= \int \frac{2\sin^2 x+1}{1+2\cos^2 x}\,\mathrm{d}x = \left|t=\tan x,\ \mathrm{d}x = \frac{1}{t^2+1}\,\mathrm{d}t\right| = \int \frac{2\frac{t^2}{1+t^2}+1}{1+2\frac{1}{1+t^2}}\cdot\frac{1}{t^2+1}\,\mathrm{d}t = \\ = \int \frac{2t^2+1+t^2}{1+t^2}\cdot\frac{1+t^2}{1+t^2+2}\cdot\frac{1}{t^2+1}\,\mathrm{d}t &= \int \frac{3t^2+1}{(t^2+3)(t^2+1)}\,\mathrm{d}t\end{aligned}$$

Parciální zlomky:

$$\begin{aligned}&\frac{3t^2+1}{(t^2+3)(t^2+1)} = \frac{At+B}{t^2+3} + \frac{Ct+D}{t^2+1}\\ &3t^2+1 = (At+B)(t^2+1) + (Ct+D)(t^2+3)\\ &3t^2+1 = At^3+At+Bt^2+B+Ct^3+3Ct+Dt^2+3D\end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}&0 = A+C\\ &3 = B+D \overset{4}{\Longrightarrow} 3 = B-1 \Longrightarrow B=4\\ &0 = A+3C \overset{1}{\Longrightarrow} 0 = 2C \Longrightarrow C=0 \overset{1}{\Longrightarrow} A=0\\ &1 = B+3D \overset{2}{\Longrightarrow} 1 = 3+2D \Longrightarrow 2D=-2 \Longrightarrow D=-1\end{aligned}$$

$$\int \frac{3t^2+1}{(t^2+3)(t^2+1)}\,\mathrm{d}t \overset{\text{lin.}}{=} 4\int \frac{1}{t^2+3}\,\mathrm{d}t - \int \frac{1}{t^2+1}\,\mathrm{d}t = \frac{4}{3}\int \frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2+1}\,\mathrm{d}t - \arctan t =$$
$$= \left| u = \frac{t}{\sqrt{3}},\ \mathrm{d}u = \frac{1}{\sqrt{3}}\,\mathrm{d}t \right| = \frac{4\sqrt{3}}{3}\int \frac{1}{y^2+1}\,\mathrm{d}y - \arctan t \overset{\text{c}}{=} \frac{4\sqrt{3}}{3}\arctan y - \arctan t =$$
$$\overset{\text{c}}{=} \frac{4\sqrt{3}}{3}\arctan\frac{t}{\sqrt{3}} - \arctan t \overset{\text{c}}{=} \frac{4\sqrt{3}}{3}\arctan\frac{\tan x}{\sqrt{3}} - \arctan\tan x \overset{\text{c}}{=} \frac{4\sqrt{3}}{3}\arctan\frac{\tan x}{\sqrt{3}} - x$$

**Příklad 3 (a)** $\int \frac{\log x}{x - x\log x}\,\mathrm{d}x$
Podmínky: $x > 0, x \neq 0, x \neq e$.

$$\int \frac{\log x}{x - x\log x}\,\mathrm{d}x = \int \frac{1}{x}\frac{\log x}{1-\log x}\,\mathrm{d}x = \left| y = \log x,\ \mathrm{d}y = \frac{1}{x}\,\mathrm{d}x \right| = \int \frac{y}{1-y}\,\mathrm{d}y =$$
$$= \int \frac{y-1+1}{1-y}\,\mathrm{d}y \overset{\text{lin.}}{=} \int -1\,\mathrm{d}y + \int \frac{-1}{y-1}\,\mathrm{d}y = |u = y-1,\ \mathrm{d}u = \mathrm{d}y| = -y - \int \frac{1}{u}\,\mathrm{d}u =$$
$$\overset{\text{c}}{=} -y - \log|u| \overset{\text{c}}{=} -y - \log|y-1| \overset{\text{c}}{=} -\log x - \log|\log x - 1|$$

**Příklad 3 (b)** $\int \frac{1}{e^{2x}+e^x-2}\,\mathrm{d}x$
Podmínky: $e^{2x}+e^x-2 = (e^x-1)(e^x+2) \neq 0 \implies x \neq 0.$

$$\int \frac{1}{e^{2x}+e^x-2}\,\mathrm{d}x = \int \frac{1}{(e^x)^2+e^x-2}\,\mathrm{d}x = |y = e^x,\ \mathrm{d}y = e^x\,\mathrm{d}x| = \int \frac{1}{y(y^2+y-2)}\,\mathrm{d}y$$

Parciální zlomky:

$$\frac{1}{y(y^2+y-2)} = \frac{1}{y(y+2)(y-1)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-1}$$
$$1 = A(y^2+y-2) + By(y-1) + Cy(y+2)$$
$$y = 0 \implies 1 = -2A \implies A = -\frac{1}{2}$$
$$y = -2 \implies 1 = -2B(-2-1) \implies 1 = 6B \implies B = \frac{1}{6}$$
$$y = 1 \implies 1 = 3C \implies C = \frac{1}{3}$$

$$\int \frac{1}{y(y^2+y-2)}\,\mathrm{d}y \overset{\text{lin.}}{=} -\frac{1}{2}\int \frac{1}{y}\,\mathrm{d}y + \frac{1}{6}\int \frac{1}{y+2}\,\mathrm{d}y + \frac{1}{3}\int \frac{1}{y-1}\,\mathrm{d}y =$$
$$\overset{\text{c}}{=} -\frac{1}{6}\log|y| + \frac{1}{6}\log|y+2| + \frac{1}{3}\log|y-1| =$$
$$\overset{\text{c}}{=} -\frac{1}{2}\log(e^x) + \frac{1}{6}\log(e^x+2) + \frac{1}{3}\log|e^x-1| \overset{\text{c}}{=} -\frac{1}{2}x + \frac{1}{6}\log(e^x+2) + \frac{1}{3}\log|e^x-1|$$

Poznámka: symbolem log máme na mysli přirozený logaritmus.

**Příklad 3 (c)** $\int \frac{\sqrt[4]{x}}{\sqrt[4]{x}+\sqrt{x}}\,\mathrm{d}x$

Podmínky: $x > 0$.

$$\int \frac{\sqrt[4]{x}}{\sqrt[4]{x}+\sqrt{x}}\,\mathrm{d}x = \left| y=\sqrt[4]{x},\ \mathrm{d}y = \frac{1}{4}x^{-\frac{3}{4}}\,\mathrm{d}x \right| = \int \frac{y}{y+y^2}\cdot 4y^3\,\mathrm{d}y \overset{\text{lin.}}{=} 4\int \frac{y^3}{y+1}\,\mathrm{d}y =$$
$$= 4\int y^2 - y + 1 - \frac{1}{y+1}\,\mathrm{d}y \overset{\text{c}}{=} \frac{4}{3}y^3 - 2y^2 + 4y - \log|y+1| \overset{\text{c}}{=} \frac{4}{3}\sqrt{x^3} - 2\sqrt{x} + 4\sqrt[4]{x} - \log|\sqrt[4]{x}+1|$$

**Příklad 3 (d)** $\int \frac{\sqrt{2x+3}+x}{\sqrt{2x+3}-x}\,\mathrm{d}x$

Podmínky: $2x+3 \geq 0 \implies x \geq -\frac{3}{2}$ a $2x+3-x^2 = 0 \implies x \neq 3, x \neq -1$. Vnitřek integrálu je tedy definován na $\left[-\frac{3}{2},-1\right) \cup (-1,3) \cup (0,\infty)$. Má tedy smyl integrál uvažovat na intervalech $\left(-\frac{3}{2},-1\right), (-1,3), (3,\infty)$.

$$\int \frac{\sqrt{2x+3}+x}{\sqrt{2x+3}-x}\,\mathrm{d}x = \left| y=\sqrt{2x+3}, \frac{y^2-3}{2} = x,\ \mathrm{d}y = \frac{1}{\sqrt{2x+3}}\,\mathrm{d}x \right| = \int \frac{y+\frac{y^2-3}{2}}{y-\frac{y^2-3}{2}}\cdot y\,\mathrm{d}y =$$
$$= \int \frac{2y+y^2-3}{2y-y^2+3}y\,\mathrm{d}y = \int \frac{-y^3-2y^2+3y}{y^2-2y-3}\,\mathrm{d}y = \int -y-4+\frac{-8y-12}{y^2-2y-3}\,\mathrm{d}y$$

Parciální zlomky:

$$\frac{-8y-12}{y^2-2y-3} = \frac{-8y-12}{(y-3)(y+1)} = \frac{A}{y-3} + \frac{B}{y+1}$$
$$-8y-12 = A(y+1) + B(y-3)$$
$$y=-1 \implies 8-12 = -4B \implies B = 1$$
$$y=3 \implies -24-12 = 4A \implies A = -\frac{36}{4} = -9$$

$$\int -y-4+\frac{-8y-12}{y^2-2y-3}\,\mathrm{d}y \overset{\text{lin.}}{=} -\frac{y^2}{2} - 4y - 9\int \frac{1}{y-3}\,\mathrm{d}y + \int \frac{1}{y+1}\,\mathrm{d}y =$$
$$\overset{\text{c}}{=} -\frac{y^2}{2} - 4y - 9\log|y-3| + \log|y+1| =$$
$$\overset{\text{c}}{=} -\frac{2x+3}{2} - 4\sqrt{2x+3} - 9\log|\sqrt{2x+3}-3| + \log\left(\sqrt{2x+3}+1\right)$$

**Příklad 3 (e)** $\int \frac{e^{4x}+e^{2x}}{e^{3x}-1}\,\mathrm{d}x$

Podmínky: $e^{3x} \neq 1 \implies 3x \neq 0 \implies x \neq 0$

$$\int \frac{e^{4x}+e^{2x}}{e^{3x}-1}\,\mathrm{d}x = |y=e^x,\ \mathrm{d}y = e^x\,\mathrm{d}x| = \int \frac{y^3+y}{y^3-1}\,\mathrm{d}y = \int \frac{y^3-1+y+1}{y^3-1}\,\mathrm{d}y =$$
$$\overset{\text{lin.}}{=} y + \int \frac{y+1}{y^3-1}\,\mathrm{d}y$$

Parciální zlomky:

$$\frac{y+1}{y^3-1} = \frac{y+1}{(y-1)(y^2+y+1)} = \frac{A}{y-1} + \frac{By+C}{y^2+y+1}$$
$$y+1 = A(y^2+y+1) + (By+C)(y-1)$$
$$y=1 \implies 2 = 3A \implies A = \frac{2}{3}$$
$$y+1 = \frac{2}{3}y^2 + \frac{2}{3}y + \frac{2}{3} + By^2 - By + Cy - C$$
$$y+1 = y^2\left(\frac{2}{3}+B\right) + y\left(\frac{2}{3} - B + C\right) + \frac{2}{3} - C$$

Porovnáním koeficientů dostáváme:

$$0 = \frac{2}{3} + B \implies B = -\frac{2}{3}$$
$$1 = \frac{2}{3} - B + C$$
$$1 = \frac{2}{3} - C \implies C = \frac{2}{3} - 1 = -\frac{1}{3}$$

Máme tedy: $A = \frac{2}{3}, B = -\frac{2}{3}, C = -\frac{1}{3}$.

$$y + \int \frac{y+1}{y^3-1}\ \mathrm{d}y = y + \frac{2}{3}\int \frac{1}{y-1}\ \mathrm{d}y - \frac{1}{3}\int \frac{2y+1}{y^2+y+1} = \left|u = y^2+y+1,\ \mathrm{d}u = (2y+1)\ \mathrm{d}y\right| =$$
$$= y + \frac{2}{3}\log|y-1| - \frac{1}{3}\int \frac{1}{u}\ \mathrm{d}u \overset{c}{=} y + \frac{2}{3}\log|y-1| - \frac{1}{3}\log|y^2+y+1| =$$
$$\overset{c}{=} e^x + \frac{2}{3}\log|e^x - 1| - \frac{1}{3}\log(e^{2x} + e^x + 1)$$

**Příklad 3 (f)** $\int \frac{1}{x\log x \cdot \log(\log x)}\ \mathrm{d}x$

$$\int \frac{1}{x\log x \cdot \log(\log x)}\ \mathrm{d}x = \left|y = \log x,\ \mathrm{d}y = \frac{1}{x}\ \mathrm{d}x\right| = \int \frac{1}{y\log(y)}\ \mathrm{d}y = \left|u = \log y,\ \mathrm{d}u = \frac{1}{y}\ \mathrm{d}y\right| =$$
$$= \int \frac{1}{u}\ \mathrm{d}u \overset{c}{=} \log|u| \overset{c}{=} \log|\log y| \overset{c}{=} \log|\log(\log x)|$$

**Příklad 3 (g)** $\int \frac{1}{x}\sqrt{\frac{x+2}{x-3}}\ \mathrm{d}x$

**Výběr substituce:** $R\left(x, \sqrt{\frac{x+2}{x-3}}\right) = \frac{1}{x}\sqrt{\frac{x+2}{x-3}}, (R(u,v) = \frac{v}{u})$

Provedeme substituci $t = \sqrt{\frac{x+2}{x-3}}$. Vyjádřeme nejdříve $x$ v závislosti na $t$.

$$t^2 = \frac{x+2}{x-3} \implies t^2(x-3) = x+2 \implies x(t^2-1) = 2 + 3t^2 \implies x = \frac{3t^2+2}{t^2-1}$$

Zderivováním získaného vztahu dostaneme vztah pro $\mathrm{d}x$ a $\mathrm{d}t$:

$$\mathrm{d}x = \frac{6t(t^2-1) - (3t^2+2)(2t)}{(t^2-1)^2}\ \mathrm{d}t = \frac{6t^3 - 6t - 6t^3 - 4t}{(t^2-1)^2}\ \mathrm{d}t = \frac{-10t}{(t^2-1)^2}\ \mathrm{d}t$$

$$\int \frac{1}{x}\sqrt{\frac{x+2}{x-3}}\,\mathrm{d}x = \left| t=\sqrt{\frac{x+2}{x-3}}, x=\frac{3t^2+2}{t^2-1},\ \mathrm{d}x=\frac{-10t}{(t^2-1)^2}\,\mathrm{d}t \right| = \int \frac{t^2-1}{3t^2+2}\cdot t\cdot \frac{-10t}{(t^2-1)^2}\,\mathrm{d}t =$$
$$= \int \frac{-10t^2}{(t^2-1)(3t^2+2}\,\mathrm{d}t$$

Parciální zlomky:

$$\begin{aligned}
&\frac{-10t^2}{(t^2-1)(3t^2+2)} = \frac{-10t^2}{(t-1)(t+1)(3t^2+2)} = \frac{A}{t-1}+\frac{B}{t+1}+\frac{Ct+D}{3t^2+2}\\
&-10t^2 = (At+A)(3t^2+2)+(Bt-B)(3t^2+2)+(Ct+D)(t^2-1)\\
t=1 \implies &-10 = 2A(3+2) \implies A=-1\\
t=-1 \implies &-10 = -10B \implies B=1\\
&-10t^2 = -3t^3-2t-3t^2-2+3t^3+2t-3t^2-2+Ct^3-Ct+Dt^2-D\\
&-10t^2 = Ct^3+(D-6)t^2-Ct-4-D
\end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}
&C=0\\
&-10 = D-6 \implies D=-4\\
&0=-C\\
&0=-4-D
\end{aligned}$$

Řešení soustavy: $A=-1, B=1, C=0, D=-4$.

$$\begin{aligned}
\int \frac{-10t^2}{(t^2-1)(3t^2+2}\,\mathrm{d}t &\overset{\text{lin.}}{=} -\int\frac{1}{t-1}\,\mathrm{d}t+\int\frac{1}{t+1}\,\mathrm{d}t-4\int\frac{1}{3t^2+2}\,\mathrm{d}t =\\
&\overset{\text{c}}{=} -\log|t-1|+\log|t+1|-\frac{2\sqrt{2}}{\sqrt{3}}\arctan\sqrt{\frac{3}{2}}t =\\
&\overset{\text{c}}{=} -\log\left|\sqrt{\frac{x+2}{x-3}}-1\right|+\log\left|\sqrt{\frac{x+2}{x-3}}+1\right|-\frac{2\sqrt{2}}{\sqrt{3}}\arctan\left(\sqrt{\frac{3}{2}}\sqrt{\frac{x+2}{x-3}}\right)
\end{aligned}$$

Poslední integrály jsme spočetli jako již mnohokrát dříve - substitucí a úpravou výrazu...

**Příklad 3 (h)** $\int \frac{1-\sqrt{x+1}}{1-\sqrt[3]{x+1}}\,\mathrm{d}x$

**Výběr substituce:**
Převedeme nejdříve všechny odmocniny na mocniny též odmocniny: $\sqrt{x}=(\sqrt[6]{x})^3$, $\sqrt[3]{x}=(\sqrt[6]{x})^2$
$R\,(x,\sqrt[6]{x}) = \frac{1-\sqrt{x+1}}{1-\sqrt[3]{x+1}}, (R(u,v)=\frac{1-v^3}{1-v^2})$

$$\int \frac{1-\sqrt{x+1}}{1-\sqrt[3]{x+1}}\ \mathrm{d}x = \left| t=\sqrt[6]{x+1}, t^3=\sqrt{x+1}, t^2=\sqrt[3]{x+1}, x=t^6-1 \implies \mathrm{d}x = 6t^5\ \mathrm{d}t \right| =$$

$$= \int \frac{1-t^3}{1-t^2}6t^5\ \mathrm{d}t = \int \frac{6t^5(t-1)(t^2+t+1)}{(t+1)(t-1)}\ \mathrm{d}t = \int \frac{6t^7+6t^6+6t^5}{t+1}\ \mathrm{d}t =$$

$$= \int 6t^6+6t^4-6t^3+6t^2-6t+6-\frac{6}{t+1}\ \mathrm{d}t =$$

$$\stackrel{c}{=} \frac{6}{7}t^7+\frac{6}{5}t^5-\frac{6}{4}t^4+\frac{6}{3}t^3-\frac{6}{2}t^2+6t-6\log|t+1| =$$

$$\stackrel{c}{=} \frac{6}{7}\sqrt[6]{(x+1)^7}+\frac{6}{5}\sqrt[6]{(x+1)^5}-\frac{3}{2}\sqrt[6]{(x+1)^4}+2\sqrt[6]{(x+1)^3}-3\sqrt[6]{(x+1)^2}+6\sqrt[6]{x+1}-$$
$$-6\log(\sqrt[6]{x+1}+1)$$

**Příklad 3 (i)** $\int \frac{1}{x(\log^3 x-1)}\ \mathrm{d}x$

$$\int \frac{1}{x(\log^3 x-1)}\ \mathrm{d}x = \left| y=\log x,\ \mathrm{d}y=\frac{1}{x}\ \mathrm{d}x \right| = \int \frac{1}{y^3-1}\ \mathrm{d}y$$

Parciální zlomky:

$$\frac{1}{y^3-1} = \frac{1}{(y-1)(y^2+y+1)} = \frac{A}{y-1}+\frac{By+C}{y^2+y+1}$$
$$1 = Ay^2+Ay+A+By^2-By+Cy-C$$

Porovnáním koeficientů dostáváme:

$$0 = A+B$$
$$0 = A-B+C$$
$$1 = A-C$$

Řešení soustavy: $A=\frac{1}{3}, B=-\frac{1}{3}, C=-\frac{2}{3}$.

$$\int \frac{1}{y^3-1}\ \mathrm{d}y \stackrel{\text{lin.}}{=} \frac{1}{3}\int \frac{1}{y-1}\ \mathrm{d}y - \frac{1}{3}\int \frac{y+2}{y^2+y+1}\ \mathrm{d}y = \left| u=y^2+y+1,\ \mathrm{d}u=(2y+1)\ \mathrm{d}y \right| =$$

$$= \frac{1}{3}\log|y-1| - \frac{1}{6}\int \frac{2y+4-3+3}{y^2+y+1}\ \mathrm{d}y = \frac{1}{3}\log|y-1| - \frac{1}{6}\int \frac{1}{u}\ \mathrm{d}u - \frac{1}{2}\int \frac{1}{y^2+y+1}\ \mathrm{d}y =$$

$$= \frac{1}{3}\log|y-1| - \frac{1}{6}\log|u| - \frac{1}{2}\int \frac{1}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}\ \mathrm{d}y = \frac{1}{3}\log|y-1| - \frac{1}{6}\log|u| - \frac{2}{3}\int \frac{1}{\left(\frac{y+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\ \mathrm{d}y =$$

$$\stackrel{c}{=} \frac{1}{3}\log|y-1| - \frac{1}{6}\log|u| - \frac{2}{3}\sqrt{\frac{3}{4}}\arctan\left(\frac{y+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right) =$$

$$\stackrel{c}{=} \frac{1}{3}\log|y-1| - \frac{1}{6}\log|y^2+y+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\left(y+\frac{1}{2}\right)\right) =$$

$$\stackrel{c}{=} \frac{1}{3}\log|\log x-1| - \frac{1}{6}\log|\log^2 x+\log x+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(\log x+\frac{1}{2}\right)\right)$$

**Příklad 3 (j)** $\int \sqrt{\frac{x-1}{x+2}}\,\mathrm{d}x$

**Výběr substituce:** $R\left(x, \sqrt{\frac{x-1}{x+2}}\right) =, (R(u,v) = v)$

Provedeme substituci: $t = \sqrt{\frac{x-1}{x+2}}$. Odvoďme $x,\ \mathrm{d}x$.

$$t^2 = \frac{x-1}{x+2} \implies t^2(x+2) = x-1 \implies x(t^2-1) = -1-2t^2 \implies x = \frac{-2t^2-1}{t^2-1}$$

$$x = \frac{-2t^2-1}{t^2-1} \implies \mathrm{d}x = \frac{-4t(t^2-1)-(-2t^2-1)2t}{(t^2-1)^2}\,\mathrm{d}t = \frac{-4t^3+4t+4t^3+2t}{(t^2-1)^2}\,\mathrm{d}t = \frac{6t}{(t^2-1)^2}\,\mathrm{d}t$$

$$\int \sqrt{\frac{x-1}{x+2}}\,\mathrm{d}x = \left| t = \sqrt{\frac{x-1}{x+2}},\ \mathrm{d}x = \frac{6t}{(t^2-1)^2}\,\mathrm{d}t \right| = \int \frac{6t^2}{(t^2-1)^2}\,\mathrm{d}t$$

Parciální zlomky:

$$\begin{aligned}
\frac{6t^2}{(t^2-1)^2} &= \frac{6t^2}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2} \\
6t^2 &= (At-A+B)(t^2+2t+1) + (Ct+C+D)(t^2-2t+1) \\
t = 1 \implies 6 &= 4B \implies B = \frac{3}{2} \\
t = -1 \implies 6 &= 4D \implies D = \frac{3}{2} \\
6t^2 &= At^3 + 2At^2 + At - At^2 - 2At - A + \frac{3}{2}t^2 + 3t + \frac{3}{2} + Ct^3 - 2Ct^2 + Ct + Ct^2 - 2Ct + \\
&\quad + C + \frac{3}{2}t^2 - 3t + \frac{3}{2}
\end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}
0 &= A + C \implies A = -C \overset{2}{\implies} A = \frac{3}{2} \\
6 &= A + 3 - C \overset{1}{\implies} 3 = -2C \implies C = -\frac{3}{2} \\
0 &= -A - C \\
0 &= -A + 3 + C
\end{aligned}$$

Řešením soustavy je: $A = \frac{3}{2}, B = \frac{3}{2}, C = -\frac{3}{2}, D = \frac{3}{2}$.

$$\begin{aligned}
&\int \frac{6t^2}{(t^2-1)^2}\,\mathrm{d}t \overset{\text{lin.}}{=} \frac{3}{2}\int \frac{1}{t-1}\,\mathrm{d}t + \frac{3}{2}\int \frac{1}{(t-1)^2}\,\mathrm{d}t - \frac{3}{2}\int \frac{1}{t+1}\,\mathrm{d}t + \frac{3}{2}\int \frac{1}{(t+1)^2}\,\mathrm{d}t = \\
&\overset{c}{=} \frac{3}{2}\log|t-1| - \frac{3}{2}\frac{1}{t+1} - \frac{3}{2}\log|t+1| - \frac{3}{2}\frac{1}{t+1} = \\
&\overset{c}{=} \frac{3}{2}\log\left|\sqrt{\frac{x-1}{x+2}} - 1\right| - \frac{3}{2}\frac{1}{\sqrt{\frac{x-1}{x+2}}+1} - \frac{3}{2}\log\left|\sqrt{\frac{x-1}{x+2}} + 1\right| - \frac{3}{2}\frac{1}{\sqrt{\frac{x-1}{x+2}}+1}
\end{aligned}$$

**Příklad 3 (k)** $\int \frac{1}{x}\sqrt{x^2-2x}\,\mathrm{d}x$

**Výběr substituce:** $R\left(x, \sqrt{x^2-2x}\right) = \frac{1}{x}\sqrt{x^2-2x}, (R(u,v) = \frac{v}{u})$

Platí, že $x^2 - 2x = x(x-2)$. Provedeme substituci $t = \sqrt{\frac{x}{x-2}}$ (dle nápovědy z teorie k 10. cvičení).

$$t^2 = \frac{x}{x-2} \implies t^2x - 2t^2 = x \implies x = \frac{2t^2}{t^2-1} \implies \mathrm{d}x = \frac{-4t}{(t^2-1)^2}$$

$$\int \frac{1}{x}\sqrt{x^2-2x}\,\mathrm{d}x = \left| t = \sqrt{\frac{x}{x-2}},\ \mathrm{d}x = \frac{-4t}{(t^2-1)^2} \right| = \int \frac{t^2-1}{2t^2}\sqrt{\left(\frac{2t^2}{t^2-1}\right)^2 - 2\frac{2t^2}{t^2-1}}\frac{-4t}{(t^2-1)^2}\,\mathrm{d}t =$$
$$= \int \frac{-2}{t(t^2-1)}\sqrt{\frac{4t^4-4t^2(t^2-1)}{(t^2-1)^2}}\,\mathrm{d}t = \int \frac{4t}{t(t^2-1)^2}\,\mathrm{d}t = \int -\frac{1}{t+1} - \frac{1}{(t+1)^2} + \frac{1}{t-1} - \frac{1}{(t-1)^2}\,\mathrm{d}t =$$
$$\stackrel{c}{=} -\log|t+1| + \frac{1}{t+1} + \log|t-1| + \frac{1}{t-1} =$$
$$\stackrel{c}{=} -\log\left|\sqrt{\frac{x}{x-2}}+1\right| + \frac{1}{\sqrt{\frac{x}{x-2}}+1} + \log\left|\sqrt{\frac{x}{x-2}}-1\right| + \frac{1}{\sqrt{\frac{x}{x-2}}-1}$$

**Příklad 3 (l)** $\int \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})}\,\mathrm{d}x$

**Výběr substituce:**
Opět převedeme na mocniny stejné odmocniny.
$R(x, \sqrt[6]{x}) = \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})}, (R(u,v) = \frac{1}{u(1+2v^3+v^2})$

$$\int \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})}\,\mathrm{d}x = \left|t = \sqrt[6]{x}, t^3 = \sqrt{x}, t^2 = \sqrt[3]{x}, x = t^6 \implies \mathrm{d}x = 6t^5\,\mathrm{d}t\right| =$$
$$\int \frac{1}{t^6\,(1+2t^3+t^2)}6t^5\,\mathrm{d}t = \int \frac{6}{t(2t^3+t^2+1)}\,\mathrm{d}t$$

Polynom $2t^3+t^2+1$ má zřejmě kořen $-1$. Z toho plyne následující.

$$\frac{6}{t(2t^3+t^2+1)} = \frac{6}{t(t+1)(2t^2-t+1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{2t^2-t+1}$$
$$6 = (At+A)(2t^2-t+1) + Bt(2t^2-t+1) + (Ct+D)(t^2+t)$$
$$t = 0 \implies 6 = A$$
$$t = -1 \implies 6 = -4B \implies B = -\frac{3}{2}$$
$$6 = 12t^3 - 6t^2 + 6t + 12t^2 - 6t + 6 - 3t^3 - \frac{3}{2}t^2 + \frac{3}{2}t + Ct^3 + Ct^2 + Dt^2 + Dt$$

Porovnáním koeficientů dostáváme:

$$
\begin{aligned}
&0 = 9 + C \implies C = -9 \\
&0 = \frac{15}{2} + C + D \overset{1}{\implies} 9 - \frac{15}{2} = D \implies D = \frac{3}{2} \\
&0 = -\frac{3}{2} + D \\
&6 = 6
\end{aligned}
$$

Řešení: $A = 6, B = -\frac{3}{2}, C = -9, D = \frac{3}{2}$

$$
\begin{aligned}
\int \frac{6}{t(2t^3+t^2+1)}\,\mathrm{d}t &\overset{\text{lin.}}{=} 6\int \frac{1}{t}\,\mathrm{d}t - \frac{3}{2}\int \frac{1}{t+1}\,\mathrm{d}t + \int \frac{-9t+\frac{3}{2}}{2t^2-t+1}\,\mathrm{d}t = \\
&= \left| y = 2t^2 - t + 1,\ \mathrm{d}y = (4t-1)\,\mathrm{d}t \right| = \\
&= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\int \frac{4t - \frac{2}{3} - \frac{1}{3} + \frac{1}{3}}{2t^2-t+1}\,\mathrm{d}t = \\
&= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\int \frac{1}{y}\,\mathrm{d}y - \frac{9}{4}\cdot\frac{1}{3}\int \frac{1}{2t^2-t+1}\,\mathrm{d}t = \\
&= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{3}{8}\int \frac{1}{t^2 - \frac{1}{2}t + \frac{1}{2}}\,\mathrm{d}t = \\
&= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{3}{8}\int \frac{1}{\left(t-\frac{1}{4}\right)^2 + \frac{7}{16}}\,\mathrm{d}t = \\
&= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{3}{8}\cdot\frac{16}{7}\int \frac{1}{\left(\frac{t-\frac{1}{4}}{\sqrt{\frac{7}{16}}}\right)^2 + 1}\,\mathrm{d}t = \\
&= \left| z = \frac{4}{\sqrt{7}}\left(t - \frac{1}{4}\right),\ \mathrm{d}z = \frac{4}{\sqrt{7}}\,\mathrm{d}t \right| = \\
&= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{6}{7}\cdot\frac{\sqrt{7}}{4}\int \frac{1}{z^2+1}\,\mathrm{d}z = \\
&\overset{\text{c}}{=} 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{3}{2\sqrt{7}}\arctan z = \\
&\overset{\text{c}}{=} 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|2t^2-t+1| - \frac{3}{2\sqrt{7}}\arctan\frac{4t-1}{\sqrt{7}} = \\
&\overset{\text{c}}{=} 6\log\sqrt[6]{x} - \frac{3}{2}\log\left(\sqrt[6]{x}+1\right) - \frac{9}{4}\log|2\sqrt[3]{x} - \sqrt[6]{x} + 1| - \frac{3}{2\sqrt{7}}\arctan\frac{4\sqrt[6]{x}-1}{\sqrt{7}}
\end{aligned}
$$